Metric Spaces and Topology
Lecture 27

Def. A *directed set* is a pair $(A, \leq)$, where A is a set and $\leq$ is a binary relation on A satisfying:

partial order
(i) $\alpha \leq \alpha \quad \forall \alpha \in A$ (reflexive)
(ii) $\alpha \leq \beta$ and $\beta \leq \gamma \Rightarrow \alpha \leq \gamma \quad \forall \alpha, \beta, \gamma$ (transitive)
(iii) $\alpha \leq \beta$ and $\beta \leq \alpha \Rightarrow \alpha = \beta \quad \forall \alpha, \beta$ (antisymmetric)

(iv) $\forall \alpha, \beta \in A$, $\exists \gamma \in A$ s.t. $\alpha \leq \gamma$ and $\beta \leq \gamma$. (has joins)

Examples. (a) Fix a set I, and let $A := \mathcal{P}_{fin}(I) :=$ the set of all finite subsets of I and let $\leq$ on A be the inclusion $\subseteq$. Indeed $\subseteq$ is a partial order and for any $\alpha, \beta \in A$, $\gamma := \alpha \cup \beta$ satisfies $\alpha \subseteq \gamma$ and $\beta \subseteq \gamma$.

(b) Let X be a top space, $x \in X$, and let $A :=$ a neighbourhood basis at x. Let $\leq$ on A be the reverse-inclusion $\supseteq$. Then $(A, \supseteq)$ is a directed set. Indeed, $\supseteq$ is a partial and $\forall\, U, V \in A$, there is $W \in A$ with $W \subseteq U \cap V$ because A is a neighb. basis.

(c) $(\mathbb{N}, \leq)$ is a directed set where $\leq$ is the usual order.

(d) If $(A, \leq_A)$ and $(B, \leq_B)$ are directed sets, then so is $(A \times B, \leq)$ where $(\alpha, \beta) \leq (\alpha', \beta') :\Leftrightarrow \alpha \leq_A \alpha'$ and $\beta \leq_B \beta'$.

Def. For a top space X, a *net* is a sequence $(x_\alpha)_{\alpha \in A}$ whose index set $(A, \leq)$ is directed. For a set $U \subseteq X$, we say that $(x_\alpha)_{\alpha \in A}$ is *eventually* in U if $\exists \alpha_0 \in A$ s.t. $\forall \alpha \geq \alpha_0$ $x_\alpha \in U$; denote this by $\forall^\infty \alpha \in A$ $x_\alpha \in U$. We say that $(x_\alpha)_{\alpha \in A}$ is *frequently* in U if $\forall \alpha_0 \in A$ $\exists \alpha \geq \alpha_0$ $x_\alpha \in U$; denote this by $\exists^\infty \alpha \in A$ $x_\alpha \in U$. We say that $(x_\alpha)_{\alpha \in A}$ converges to $x \in X$, and write $x_\alpha \to x$ or $\lim_\alpha x_\alpha = x$, if $\forall$ open $U \ni x$ $\forall^\infty \alpha$ $x_\alpha \in U$.

For example, sequences are nets and the notions of convergence coincide. We now prove statements about nets, which show that nets are a good replacement for sequences in general top. spaces.

Prop. Let X be a top. space and $Y \subseteq X$. A point $x \in \overline{Y}$ if and only if $\exists$ net $(y_\alpha) \subseteq Y$ converging to x.

Proof. $\Leftarrow$. Is immediate since then every neighbourhood of x contains a point in Y.

$\Rightarrow$. Let $x \in \overline{Y}$. Let A be a neighbourhood basis at x closed under finite intersections and let $\leq$ be the reverse-inclusion as in Example (b) above.
Then for each $U \in A$, let $x_U \in U \cap Y$ which exists since $x \in \overline{Y}$ (using AC). We show that $(x_U)_{U \in A} \to x$.
Indeed, $\forall\, U \in A$ we need to show that $\forall^{\infty} V \in A$ $x_V \in U$.
But this is true for all $V \geq U$, i.e. all $V \subseteq U$. $\square$

Prop. Let X, Y be top. spaces. A function $f : X \to Y$ is continuous at $x \in X$ $\Leftrightarrow$ for any net $(x_\alpha) \to x$, $\big(f(x_\alpha)\big) \to f(x)$.

Proof. $\Rightarrow$. This is a similar proof as for sequences.

$\Leftarrow$. We show the contrapositive. Suppose that f isn't continuous at x, i.e. $\exists$ neighbourhood $V \ni f(x)$ s.t. $f^{-1}(V)$ is not a neighbourhood of x, i.e. each neighb. $U \nsubseteq f^{-1}(V)$, i.e. $\exists\, x_U \in U$ s.t. $x_U \notin f^{-1}(V)$ (AC).
Letting A be the set of all open neighbourhoods of x ordered by reverse-inclusion, we get a net $(x_U)_{U \in A}$ converging to x, but $f(x_U) \notin V$ $\forall\, U \in A$,

so $\left(f(x_u)\right)_{u\in A} \not\to f(x)$. $\square$

Def. A point $x\in X$ is called a cluster point of a net $(x_\alpha)_{\alpha\in A}$, if for each neighbourhood $U\ni x$ $\exists^\infty \alpha\in A$ $x_\alpha\in U$.

Def. A net $(y_\beta)_{\beta\in(B,\leq_B)}$ is called a subnet of a net $(x_\alpha)_{\alpha\in(A,\leq_A)}$ if there is a function $\beta\mapsto\alpha_\beta : B\to A$ such that $y_\beta = x_{\alpha_\beta}$ and $\forall\, \alpha_0\in A$ $\forall^\infty\beta$ $\alpha_\beta \geq_A \alpha_0$.

Prop. A point $x\in X$ is a cluster point of a net $(x_\alpha)_{\alpha\in(A,\leq_A)}$ if and only if $\exists$ subnet converging to x.

Proof. $\Leftarrow$. Suppose $(y_\beta)_{\beta\in(B,\leq_B)}$ is a subnet converging to x. Fix a neighbourhood $U\ni x$. Then $\forall^\infty\beta$ $y_\beta\in U$. Fix $\alpha_0\in A$. Then we know that $\forall^\infty\beta$ $\alpha_\beta\geq_A\alpha_0$, so $\forall^\infty\beta\left(x_{\alpha_\beta}\in U \text{ and } \alpha_\beta\geq\alpha_0\right)$, so $\exists^\infty\alpha$ $x_\alpha\in U$.

$\Rightarrow$. Let x be a cluster point of $(x_\alpha)_{\alpha\in A}$, i.e. for each neighb. $U\ni x$ and $\alpha_0\in A$ $\exists\alpha\geq_A\alpha_0$ with $x_\alpha\in U$. Let $B:=(\mathcal{U}\times A, \leq)$ be the product of directed sets $(A,\leq_A)$ and $(\mathcal{U},\supseteq)$, where $\mathcal{U}$ is the set of all open neighb. of x. We build a subnet as follows: for each $(U, \alpha_0) \in B$, $\exists\, \alpha \geq_A \alpha_0$ s.t. $x_\alpha \in U$, so take a function $(U, \alpha_0) \mapsto$ such an α, denoted by $\alpha_{(U,\alpha_0)}$ (using AC), and let $y_{(U,\alpha_0)} := x_{\alpha_{(U,\alpha_0)}}$.

<u>Claim 1.</u> $(y_{(U,\alpha_0)}) \to x$.

Pf. For any open neighbh. $U \ni x$ and any $\alpha_0 \in A$, we know that if $(V, \alpha) \geq (U, \alpha_0)$, then $y_{(V,\alpha)} \in V \subseteq U$. $\square$

<u>Claim 2.</u> $(y_{(U,\alpha)})$ is a subnet of $(x_\alpha)_{\alpha \in A}$.

Pf. We only need to check that for any $\alpha_0 \in A$, $\forall^\infty (U, \alpha_1) \in \mathcal{U} \times A$ $\alpha_{(U,\alpha_1)} \geq_A \alpha_0$.
But for any $U \in \mathcal{U}$ and all $(V, \alpha_1) \geq (U, \alpha_0)$, we have that $\alpha_{(V,\alpha_1)} \geq_A \alpha_1 \geq_A \alpha_0$. $\square$ $\boxtimes$

Finally, we prove:

<u>Theorem.</u> For a top. space, the following are equivalent:

(1) X is compact.

(2) Every net has a convergent subnet.
(3) Every net has a cluster point.

Proof. (2)$\Leftrightarrow$(3). Already proven.

(3)$\Rightarrow$(1). Let $\mathcal{C}$ be a collection of closed sets with the finite intersection property, and we assume without loss of generality that $\mathcal{C}$ is closed under finite intersections. Then $(\mathcal{C}, \supseteq)$ is a directed set and we define a net $(x_K)_{K\in\mathcal{C}}$ by choosing $x_K \in K$ (AC). It's easy to check that any cluster point x of $(x_K)_{K\in\mathcal{C}}$ is in each $K\in\mathcal{C}$, hence $\bigcap \mathcal{C} \neq \emptyset$. HW

(1)$\Rightarrow$(3). Let $(x_\alpha)_{\alpha\in A}$ be a net. For each $\alpha\in A$, let $Y_\alpha := \{x_\beta : \beta \geq \alpha\}$ and note that $\{Y_\alpha\}_{\alpha\in A}$ has the finite intersection property: indeed, for any $\alpha_1, \alpha_2, \dots, \alpha_n \in A$ $\exists\, \beta\in A$ dominating $\alpha_1, \alpha_2, \dots, \alpha_n$ (by directedness) so $Y_\beta \subseteq Y_{\alpha_i}$ $\forall i$, so $\bigcap_{i=1}^{n} Y_{\alpha_i} \neq \emptyset$. Thus, so does the collection $\{\overline{Y_\alpha}\}_{\alpha\in A}$ of closures, so $\exists\, x \in \bigcap_{\alpha\in A} \overline{Y_\alpha}$. This x can be easily seen to be a cluster point of $(x_\alpha)_{\alpha\in A}$. HW $\square$